

* Limits are used to define continuity, derivatives & Integrals

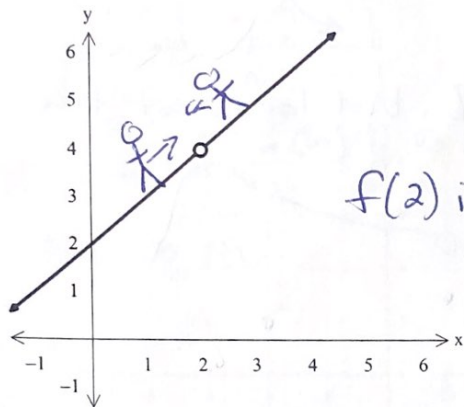
Have 1

11.1 Finding Limits Using Tables and Graphs

$$f(x) = \frac{x^2 - 4}{x - 2}$$

What happens when you approach 2?

What happens when you reach 2? You fall through the hole.



$f(2)$ is undefining

$x \rightarrow 2$ from left

$x \rightarrow 2$ from right

x	1.99	1.999	1.9999	2	2.0001	2.001	2.01
$f(x) = \frac{x^2 - 4}{x - 2}$	3.99	3.999	3.9999	undefined	4.0001	4.001	4.01

Limit Notation: $\lim_{x \rightarrow a} f(x) = L$

"height"

same equation.

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

\rightarrow as x approaches a , but remains unequal to a , the corresponding values of $f(x)$ approach L .

Using a table, find the following limits:

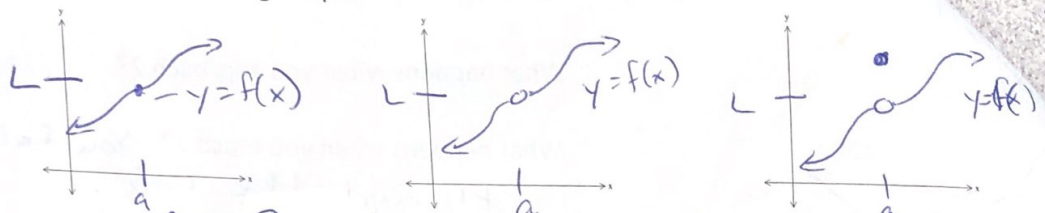
1) $\lim_{x \rightarrow 4} 3x^2 = 48$

2) $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$ Because division by 0 is undefined, the domain is $\{x \mid x \neq 0\}$. what happens as x gets close to 0?

It gets closer to 1.

3) $\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0$

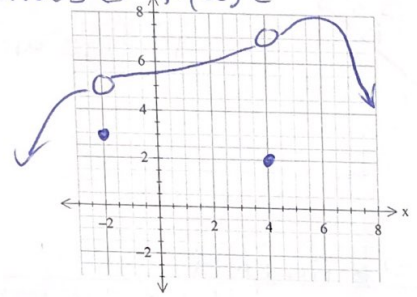
Finding Limits Using Graphs



4) Find each limit using the graph: *AS f approaches a f(a) = L, but look at the actual values @ f(a)*

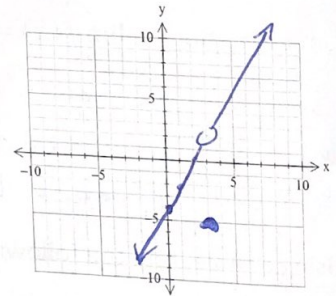
a. $\lim_{x \rightarrow 4} f(x)$
 (height @ $x=4$)
 $\lim_{x \rightarrow 4} f(x) = 7$

b. $f(4)$
 $f(4) = 2$



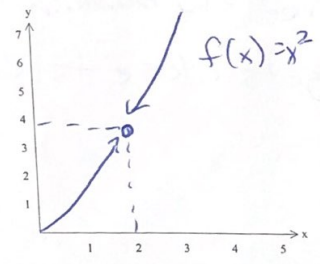
5) Find the limit by graphing the function $f(x) = \begin{cases} 2x - 4 & \text{if } x \neq 3 \\ -5 & \text{if } x = 3 \end{cases}$

$\lim_{x \rightarrow 3} f(x) = 2$ $f(3) = -5$



One Sided Limits

$\lim_{x \rightarrow 2} x^2 = 4$



x	1.999	1.9999	1.99999
$f(x) = x^2$	3.996	3.9996	3.99996

"The limit of x^2 as x approaches 2 from the left equals 4"

x	2.0001	2.001	2.01
$f(x) = x^2$	4.0004	4.004	4.0401

"The limit of x^2 as x approaches 2 from the right equals 4"

Left-Hand Limit: $\lim_{x \rightarrow a^-} f(x) = L$

Right-Hand Limit: $\lim_{x \rightarrow a^+} f(x) = L$

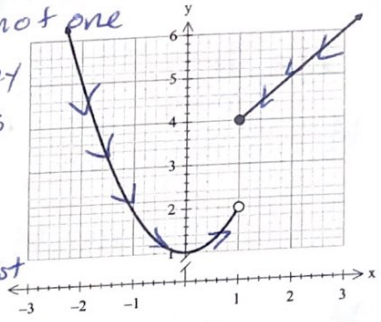
Find the one sided limits from the graphs:

$$6) f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 3 & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 4$$

★ Because there is not one single value that they both approach, f has no limit as x approaches 1 or $\lim_{x \rightarrow 1} f(x)$ does not exist

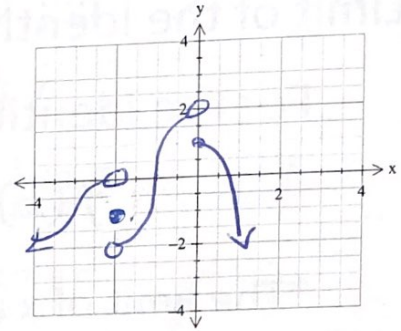


7) a. $\lim_{x \rightarrow -2^-} f(x) = 0$

b. $\lim_{x \rightarrow -2^+} f(x) = -2$

c. $\lim_{x \rightarrow -2} f(x)$ Does not exist

d. $f(-2) = -1$



8) Use the graph to find the following limits:

a. $\lim_{x \rightarrow 0^-} f(x) = 2$

b. $\lim_{x \rightarrow 0^+} f(x) = 1$

c. $\lim_{x \rightarrow 0} f(x)$ DNE

d. $f(0) = 1$

11.2 Finding Limits Using Properties of Limits

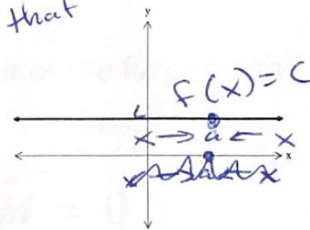
<https://www.khanacademy.org/math/ap-calculus-ab/continuity-ab/limits-of-combined-and-composite-func-ab/v/limit-properties>

Limit of a Constant Function:

Talk about "continuous" point @ that

For the constant function $f(x) = c$,

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c = c$$



*Regardless of what number x is approaching, the limit of a constant is that constant

Find the following limits:

$$1)a. \lim_{x \rightarrow 4} 7 = 7$$

$$b. \lim_{x \rightarrow 0} -5 = -5$$

Limit of the Identity Function:

For the identity function $f(x) = x$,

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} x = a$$

*The limit of x as x approaches any number is that number

Find the following limits:

$$2)a. \lim_{x \rightarrow 7} x = 7$$

$$b. \lim_{x \rightarrow -\pi} x = -\pi$$



$$\lim_{x \rightarrow 3} 7x$$

Properties of Limits:

Sum: If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$$

Difference: If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M$$

Product: $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then

$$\lim_{x \rightarrow a} [f(x) * g(x)] = \lim_{x \rightarrow a} f(x) * \lim_{x \rightarrow a} g(x) = LM$$

Limit of a Monomial: $\lim_{x \rightarrow a} b_n x^n = b_n a^n$ *The limit of a monomial as x approaches a is the monomial evaluated at a

Limit of a Polynomial: $\lim_{x \rightarrow a} f(x) = f(a)$ *The limit of a polynomial as x approaches a is the polynomial evaluated at a

Limit of a Power: $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n = L^n$

*The limit of a function to a power is found by taking the limit of the function and then raising this limit to the power

Limit of a Root: $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}$

*The limit of the nth root of a function is found by taking the limit of the function and then taking the nth root of the limit

Limit of a Quotient: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, M \neq 0$

*The limit of the quotient of two functions equals the quotient of their limits, provided that the limit of the denominator is not zero

~~Pg 1057~~

Starting
Pg. 1144

Find the following limits:

$$\lim_{x \rightarrow -4} x + \lim_{x \rightarrow -4} 9$$
$$3) \lim_{x \rightarrow -4} (x + 9)$$

$$\lim_{x \rightarrow -4} (-4 + 9) = 5$$

$$4) \lim_{x \rightarrow 5} (12 - x)$$

$$\lim_{x \rightarrow 5} (12 - 5) = 7$$

$$5) \lim_{x \rightarrow 5} (-6x)$$

$$\lim_{x \rightarrow 5} (-6 \cdot 5) = -30$$

Find the following limits:

$$6) \lim_{x \rightarrow -3} (7x - 4)$$

$$\lim_{x \rightarrow -3} (7(-3) - 4) = -25$$

$$7) \lim_{x \rightarrow 5} -6x^3$$

$$\lim_{x \rightarrow 5} -6(5)^3 = -750$$

$$8) \lim_{x \rightarrow 3} (4x^3 + 2x^2 - 6x + 5)$$

$$\lim_{x \rightarrow 3} 4(3)^3 + 2(3)^2 - 6(3) + 5$$

$$= 113$$

$$9) \lim_{x \rightarrow 5} (2x - 7)^3$$

$$\lim_{x \rightarrow 5} (2(5) - 7)^3$$

$$= 27$$

$$10) \lim_{x \rightarrow -2} \sqrt{4x^2 + 5}$$

$$\lim_{x \rightarrow -2} \sqrt{4(-2)^2 + 5}$$

$$= \sqrt{21}$$

$$11) \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 7}{2x - 5}$$

$$\lim_{x \rightarrow 1} \frac{1^3 - 3(1)^2 + 7}{2(1) - 5} = \frac{-5}{3}$$

$$12) f(x) = \begin{cases} x^2 + 5 & \text{if } x < 2 \text{ L} \\ 3x + 1 & \text{if } x \geq 2 \text{ R} \end{cases} \text{ find each of the following limits}$$

$$a. \lim_{x \rightarrow 2^-} f(x)$$

$$(2)^2 + 5 = 9$$

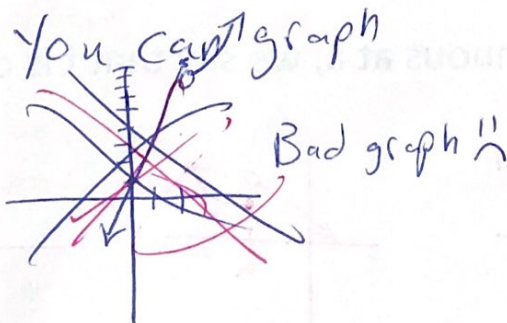
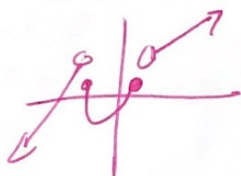
$$b. \lim_{x \rightarrow 2^+} f(x)$$

$$3(2) + 1 = 7$$

$$c. \lim_{x \rightarrow 2} f(x)$$

DNE

$$f(2) = 7$$



$$13) \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \quad \leftarrow \begin{matrix} \text{bc denominator} \\ = 0 \end{matrix}$$

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{x-3}$$

$$\lim_{x \rightarrow 3} 3 + 2 = 5$$

$$14) \lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} \quad \left(\frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2} \right)$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{4+x})^2 - 2^2}{x(\sqrt{4+x} + 2)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{4+x} + 2}$$

$$= \lim_{x \rightarrow 0} \frac{4+x-4}{x(\sqrt{4+x} + 2)} = \frac{1}{\sqrt{4+0} + 2}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

$$15) \lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x - 1}$$

$$\lim_{x \rightarrow 1} \frac{(x+3)(x-1)}{x-1}$$

$$\lim_{x \rightarrow 1} 1 + 3 = 4$$

$$16) \lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3}{x} \quad \left(\frac{\sqrt{9+x} + 3}{\sqrt{9+x} + 3} \right)$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{9+x})^2 - 3^2}{x(\sqrt{9+x} + 3)} = \frac{1}{\sqrt{9+0} + 3}$$

$$= \lim_{x \rightarrow 0} \frac{9+x-9}{x(\sqrt{9+x} + 3)} = \frac{1}{3+3} = \frac{1}{6}$$

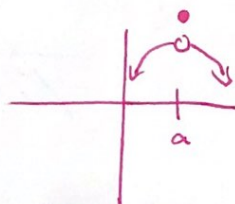
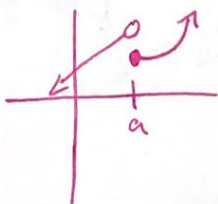
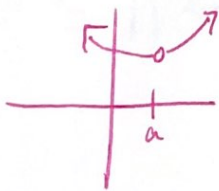
$$= \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} \sqrt{9+x} + \lim_{x \rightarrow 0} 3}$$

11.3 Limits and Continuity

The graph of a continuous function doesn't have any holes, gaps or jumps.

*Think about drawing a graph without having to lift your pencil from the paper

If f is not continuous at a , we say that f is discontinuous at a



Definition of a Function Continuous at a Number:

A function f is continuous at a when three conditions are satisfied:

1. f is defined at a (a is in the domain of f so that $f(a)$ is a real number)
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

- 1) Determine if the function $f(x) = \frac{2x+1}{2x^2-x-1}$ is continuous
- a. at 2
 - b. at 1

Check conditions for continuity with $a=2$

① Is it defined? $f(2) = \frac{4+1}{8-2-1} = 1$ real # ✓

② Does lim exist? $\frac{\lim_{x \rightarrow 2} 2(2)+1}{\lim_{x \rightarrow 2} 2(2)^2-2-1} = 1$ ✓

③ Does $\lim_{x \rightarrow 2} f(x) = f(2)$? yes ✓

Because all 3 conditions are satisfied, f is continuous at 2.

① ~~is~~ $f(1)$ defined?

If you plug in 1 \Rightarrow und.

$$\therefore \frac{2x+1}{(2x+1)(x-1)} \quad x \neq -\frac{1}{2}, 1$$

Domain $\{x \mid x \neq 1, x \neq -\frac{1}{2}\}$

$\therefore f(1)$ is discontinuous

2) Determine whether the function $f(x) = \frac{x-2}{x^2-4}$ is

continuous

a. at 1

$$\textcircled{1} f(1) = \frac{1-2}{1-4} = \frac{-2}{-3} = \frac{2}{3} \checkmark$$

$$\textcircled{2} \lim_{x \rightarrow 1} \frac{1-2}{1-3} = \frac{2}{3} \checkmark$$

$$\textcircled{3} \lim_{x \rightarrow 1} f(x) = f(1) \text{ yes } \checkmark$$

yes bc f is continuous at

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

b. at 2

$$\textcircled{1} f(2) = \frac{2-2}{4-4} = \frac{0}{0} \text{ "}$$

$$\text{so } \frac{x-2}{(x+2)(x-2)}$$

$$D: \{x \mid x \neq 2, x \neq -2\}$$

$f(2)$ is discontinuous

3) Determine for what numbers x (if any) the function is discontinuous:

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 0 \\ 2 & \text{if } 0 < x \leq 1 \\ x^2+2 & \text{if } x > 1 \end{cases}$$

★ Most functions are continuous at every # in their domain. Most of the discontinuity you encounter will be jumps due to piecewise functions.

Are they all continuous?

$f(x) = x+2 \rightarrow$ linear - continuous @ every x ,

$f(x) = 2 \rightarrow$ constant - " " "

$f(x) = x^2+2 \rightarrow$ polynomial - " " "

The piece changes at $x=0$ and $x=1$ so we must look at continuity at 0 & 1

Condition 1 for $a=0$:

Is $f(0)$ defined?

$$f(0) = 0+2 = 2 \checkmark$$

Condition 1 for $a=1$:

Is $f(1)$ defined?

$$f(1) = 2 \checkmark$$

Condition 2 for $a=0$:

Does $\lim_{x \rightarrow 0} f(x)$ exist?

$$\lim_{x \rightarrow 0^-} f(x) = 0+2 = 2 \checkmark$$

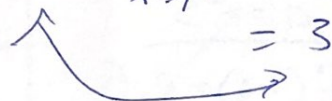
$$\lim_{x \rightarrow 0^+} f(x) = 2 \checkmark$$

Condition 2 for $a=1$:

Lim exist?

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 1^2+2 = 3$$



Not the same "

Condition 3 for $a=0$:

Does $\lim_{x \rightarrow 0} f(x) = f(0)$?

Condition 3 for $a=1$:

Function is discontinuous @ 1.

Condition 1 & 2 gave the same #, so yes the function is continuous at 0.

4) Determine for what numbers x (if any) the following function is discontinuous:

$$f(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x^2 + 1 & \text{if } 0 < x \leq 2 \\ 7 - x & \text{if } x > 2 \end{cases}$$

* changes @ 0 & 2

$a = 0$

① $f(0) = 2(0) = 0$

2.) $\lim_{x \rightarrow 0^-} 2(0) = 0$

$\lim_{x \rightarrow 0^+} 0^2 + 1 = 1$

③ $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$

Discontinuous

$a = 2$

① $f(2) = 2^2 + 1 = 5$

② $\lim_{x \rightarrow 2^-} 2^2 + 1 = 5$

$\lim_{x \rightarrow 2^+} 7 - 2 = 5$

③ $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(x)$

Continuous

5) Determine for what numbers x (if any) the following function is discontinuous:

$$f(x) = \begin{cases} \frac{x^2 + \cancel{x+3}(x-2)}{x^2 + 4x + 3 \cancel{(x+3)}(x+1)} & \text{if } x \neq -3 \\ 2.5 & \text{if } x = -3 \end{cases}$$

* changes @ -3

$a = -3$

$f(-3) = 2.5$

$\lim_{x \rightarrow -3^-} \frac{(-3-2)}{(-3+1)} = \frac{-5}{-2} = \frac{5}{2} = 2.5$

$\lim_{x \rightarrow -3^+}$ same

∴ Continuous @ -3

1.4 Intro to Derivatives

The derivative of a function of a single variable at a chosen input value, when it exists, is the slope of the tangent line to the graph of the function at that point. The tangent line is the best linear approximation of the function near that input value. For this reason, the derivative is often described as the "instantaneous rate of change", the ratio of the instantaneous change in the dependent variable to that of the independent variable.

To find the average rate of change between two points on a curved line, we find the slope of the secant line containing those points.

<https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab/secant-lines-ab/v/slope-of-a-line-secant-to-a-curve>

<https://www.khanacademy.org/math/ap-calculus-ab/derivative-introduction-ab/formal-definition-of-derivative-ab/v/alternate-form-of-the-derivative>

All of the problems we encounter in calculus make use of the derivative, so our goal in differential calculus is to become an expert at finding, or "taking," the derivative. However, before we learn a simple way to take a derivative, we must explore the "Definition of the Derivative."

The Formula:

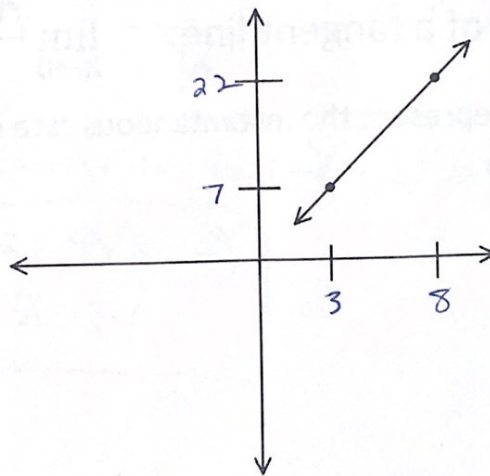
We will begin with a line and finding its slope by plugging two points into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$$\frac{22 - 7}{8 - 3} = \frac{15}{5} = 3$$

Suppose the line goes through the points (3, 7) and (8, 22). Find the slope.

The idea of "rise over run" will play an important role now. Instead of calling the x-coordinates x_1 and x_2 , we are going to call them x_1 and $x_1 + h$, where h is the difference between the two x-coordinates. (Another symbol for h is Δx , read "delta x.") Also, instead of using y_1 and y_2 , we will use $f(x_1)$ and $f(x_1 + h)$. So, now the graph looks like this:

Label the picture with the new notation.



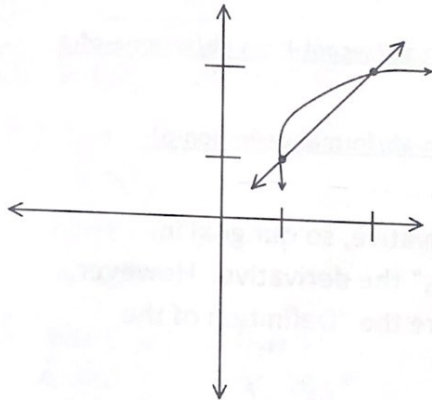
The Slope of a Curve:

Now let's look at a curve that goes through the same two points. Because it is now a curve, we cannot use the slope formula; however, we can find the approximate slope if we look at the slope of the secant line.

The equation for the slope of a secant line is: $\frac{f(x_1+h)-f(x_1)}{h}$.

This equation is called the **Difference Quotient**.

The closer the two points, the more accurate the approximation.



In fact, there is one line, called the tangent line, that touches the curve at exactly one point. The object of using the difference quotient, therefore, is to shrink h down to an infinitesimally small amount. If we could do that, then x_1 and $x_1 + h$ become the same value. We can accomplish this by using limits.

$\lim_{h \rightarrow 0} \frac{f(x_1+h)-f(x_1)}{h}$. This is the definition of the derivative, and we call it $f'(x)$.

Slope of a secant line: $\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$

*as $h \rightarrow 0$, the slope approaches the slope of the tangent line to the curve at $(a, f(a))$

Slope of a tangent line: $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$

* This represent the instantaneous rate of change of f with respect to x at a

Find the slope of the tangent line $f(x) = x^2 + x$. Then write the equation in slope intercept for m .

1) at $(2, 6)$

$$a = 2$$

$$M_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \stackrel{\rightarrow \text{plus in}}{=} \frac{[(2+h)^2 + (2+h)] - [2^2 + 2]}{h}$$

$$= \frac{\cancel{4} + 4h + h^2 + \cancel{2} + h - \cancel{4} - \cancel{2}}{h} = \frac{h^2 + 5h}{h} = \frac{h(h+5)}{h}$$

$$\lim_{h \rightarrow 0} (h+5) = 0 + 5 = 5$$

slope of tangent line at $(2, 6)$ is 5

$$\begin{aligned} y - 6 &= 5(x - 2) \\ \text{or} \\ y &= 5x - 4 \end{aligned}$$

2) at $(4, 12)$

$$a = 4$$

$$M_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \frac{[(4+h)^2 + (4+h)] - [4^2 + 4]}{h}$$

$$= \frac{\cancel{16} + 8h + h^2 + \cancel{4} + h - \cancel{16} - \cancel{4}}{h} = \frac{h^2 + 9h}{h} = h + 9$$

$$\lim_{h \rightarrow 0} 0 + 9 = 9$$

slope of tangent line @ $(4, 12) = 9$

$$\begin{aligned} y - 12 &= 9(x - 4) \\ y &= 9x - 24 \end{aligned}$$

Find the slope intercept equation of the tangent line to the graph $f(x) = \sqrt{x}$

3) at (4, 2) $a = \frac{1}{4}$

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

Need to rationalize

$$= \frac{4+h-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4} \leftarrow \text{slope}$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

4) at (1, 1) $a = \frac{1}{2}$

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$

$$= \frac{1+h-1}{h(\sqrt{1+h}+1)} = \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \leftarrow \text{slope}$$

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y - 1 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

Derivative of a Function

→ at the limit it becomes the slope of the tangent line.

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x+h) - f(x)}{h}$$

The derivative gives you a way to analyze your moving world by revealing a functions instantaneous rate of change at any moment.

5) $f(x) = x^2 + 3x$ at x

a. Find $f'(x)$

$$\lim_{h \rightarrow 0} \frac{[(x+h)^2 + 3(x+h)] - (x^2 + 3x)}{h} = \frac{x^2 + 2hx + h^2 + 3x + 3h - x^2 - 3x}{h}$$

$$\lim_{h \rightarrow 0} \frac{h^2 + 2hx + 3h}{h} = \frac{h(h + 2x + 3)}{h} = 0 + 2x + 3$$

$$f'(x) = 2x + 3$$

The derivative gives the slope of tangent line at any point

b. Find slope of tangent line at $x = -\frac{3}{2}$ and $x = -2$

$$f'(-\frac{3}{2}) = 2(-\frac{3}{2}) + 3 = -3 + 3 = 0$$

$$f'(-2) = 2(-2) + 3 = -4 + 3 = -1$$

$$6) f(x) = x^2 - 5x \text{ at } x$$

$$a. f'(x)$$

$$\lim_{h \rightarrow 0} \frac{[(x+h)^2 - 5(x+h)] - (x^2 - 5x)}{h}$$

$$= \frac{\cancel{x^2} + 2xh + h^2 - \cancel{5x} - \cancel{5h} - \cancel{x^2} + \cancel{5x}}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{h}(h + 2x - 5)}{\cancel{h}} = 0 + 2x - 5 = \boxed{2x - 5}$$

b. Find the slope of the tangent line at $x = -1$ and $x = 3$

$$f(-1) = 2(-1) - 5 = \boxed{-7}$$

$$f(3) = 2(3) - 5 = \boxed{1}$$

Instantaneous Velocity:

7) A ball is thrown straight up from a rooftop 160 ft high with an initial velocity of 48 feet per second.

The function $s(t) = -16t^2 + 48t + 160$ describes the ball's height above the ground in feet, t seconds after it is thrown. The ball misses the rooftop on its way down and eventually strikes the ground.

a. What is the instantaneous velocity of the ball 2 seconds after it is thrown? *replace t in $s(t)$ with $a+h$*

$$s'(a) = \lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \frac{[-16(a+h)^2 + 48(a+h) + 160] - [-16a^2 + 48a + 160]}{h}$$

$$\lim_{h \rightarrow 0} \frac{-16a^2 - 32ah - 16h^2 + 48a + 48h + 160 + 16a^2 - 48a - 160}{h}$$

$$\lim_{h \rightarrow 0} \frac{-16h - 32a + 48}{h} = -16(0) - 32a + 48$$
$$= -32a + 48$$

$$s'(a) = -32a + 48 \quad s'(2) = -32(2) + 48 = \boxed{-16 \text{ ft/sec}}$$

b. What is the instantaneous velocity of the ball when it hits the ground?

What time does it hit the ground?

$$0 = -16t^2 + 48t + 160$$

$$0 = -16(t-5)(t+2) = 0$$

$$t = 5, \quad \cancel{t = -2}$$

$$s'(5) = -32(5) + 48 = \boxed{-112 \text{ ft/sec}}$$

8) A ball is thrown straight up from ground level with an initial velocity of 96 feet per second. The function $s(t) = -16t^2 + 96t$ describes the ball's height above the ground in feet, t seconds after it is thrown.

a. What is the instantaneous velocity of the ball after 4 seconds?

$$s'(a) = \lim_{h \rightarrow 0} \frac{-16(a+h)^2 + 96(a+h) - (-16a^2 + 96a)}{h}$$

$$\lim_{h \rightarrow 0} \frac{-16a^2 - 32ah - 16h^2 + 96a + 96h + 16a^2 - 96a}{h}$$

$$\lim_{h \rightarrow 0} \frac{-16h - 32a + 96}{h}$$

$$\lim_{h \rightarrow 0} -16(0) - 32a + 96$$

$$s'(4) = -32(4) + 96 = \boxed{-32 \text{ ft/sec}}$$

b. What is the instantaneous velocity of the ball when it hits the ground?

$$-16t^2 + 96t = 0$$

$$-16t(t-6) = 0$$

$$t = 0, 6$$

$$s'(6) = -32(6) + 96 = \boxed{-96 \text{ ft/sec}}$$

9) Find the slope of the curve $f(x) = x^2$ at the point $(2, 4)$ $a = 2$

$$\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 4}{h} = \frac{h^2 + 4h + 4 - 4}{h} = \frac{h(h+4)}{h} = h+4$$

$$\lim_{h \rightarrow 0} (h+4) = 0+4 = 4$$

$$\boxed{\text{Slope} = 4}$$

$$a=5$$

10) Find the slope of the equation in Example 9 at the point (5, 25)

$$\lim_{h \rightarrow 0} \frac{(h+5)^2 - 5^2}{h} = \frac{h^2 + 10h + \cancel{25} - \cancel{25}}{h} = \frac{h(h+10)}{h} = h+10$$

$$\lim_{h \rightarrow 0} (0+10) = 10$$

$$\boxed{\text{slope} = 10}$$

11) Find the slope of $f(x) = x^2$ at the point (x_1, x_1^2) $a=x_1$

$$\lim_{h \rightarrow 0} \frac{(x_1+h)^2 - (x_1)^2}{h} = \frac{\cancel{x_1^2} + 2x_1h + h^2 - \cancel{x_1^2}}{h} = \frac{h(2x_1+h)}{h}$$

$$\lim_{h \rightarrow 0} (2x_1+h) = 2x_1 + 0 = 2x_1$$

$$f'(x) = 2x_1$$

12) Find the derivative of $f(x) = x^3$

$$\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$\lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2$$

$$\lim_{h \rightarrow 0} 3x^2 + 3x(0) + (0)^2 = 3x^2$$

$$\boxed{f'(x) = 3x^2}$$

13) Find the derivative of $f(x) = \sqrt{x}$. (Hint: You will need to multiply by the conjugate of the numerator.)

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{x+h} - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \text{ or } \frac{1}{2}x^{-1/2}$$

Day 1

The Power Rule:

Fortunately, there is a shortcut to taking derivatives. However, the definition of the derivative will come back in Calculus.

The basic technique for taking a derivative is called the **Power Rule**.

$$\text{If } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1}$$

This, and all other rules for derivatives can be derived from the definition of the derivative. Look at the following examples to see the power rule in action:

Example 1: If $y = x^5$, then $\frac{dy}{dx} = 5x^4$.

Example 2: If $y = x^{20}$, then $\frac{dy}{dx} = 20x^{19}$

Example 3: If $y = x^{-5}$, then $\frac{dy}{dx} = -5x^{-6}$

Example 4: Find the derivative of the function; $y = x^{\frac{1}{2}}$.

$$\frac{1}{2\sqrt{x}} \text{ or } \frac{1}{2}x^{-1/2}$$

~~The Addition Rule:~~

Nothing really changes with this rule. We now just take the derivative of each term.

Example 5: If $y = 3x^4 + 8x^{10}$, then $\frac{dy}{dx} = 12x^3 + 80x^9$.

Example 6: $y = 7x^{-4} + 5x^{-\frac{1}{2}}$, then $\frac{dy}{dx} =$

Example 7: Find the derivative of the polynomial; $y = 6x^{-7} - 4\sqrt{x}$.

The Tangent Line Problem:

One of the primary reasons we find derivatives is to identify the equation of a tangent line at a point. To begin, we must understand that we only need to identify the slope of the curve at the given point and identify the x and y values of the point. Once we have this information, we can write the equation of the line in point-slope form.

Example 8:

Find the equation of the tangent line to the function $y = x^2 - 1$ at $x = 2$.

Step 1: $\frac{dy}{dx} = 2x$. *find derivative*

Step 2: Evaluate $\frac{dy}{dx}$ at the x -value of 2 *$m = (2)(2)$ so $m = 4$*

Step 3: Identify x and y values at the point *$x = 2$, and $y = 2^2 - 1 = 3$*

Step 4: Write the equation in point-slope form *$y - 3 = 4(x - 2)$*

Example 9:

Find the equation of the tangent line to the function $y = 2x + \sqrt{x}$ at $x = 36$

$$y' = 2x + x^{1/2} = 2 + \frac{1}{2\sqrt{x}} \quad @ x=36 \quad 2 + \frac{1}{2\sqrt{36}} = 2 + \frac{1}{12} = \frac{25}{12}$$

-To find corresponding y

$$y = 2(36) + \sqrt{36} = 72 + 6 = 78$$

$$(36, 78)$$

slope @
 $x = 36$

$$y - 78 = \frac{25}{12}(x - 36)$$

Example 10: Find the points on the curve $y = 2x^3 + \frac{13}{2}x^2 + 2x + 4$ where the tangent line would be horizontal.

$$\text{slope } y' = 6x^2 + 13x + 2$$

$$\Rightarrow 0 = (6x + 1)(x + 2)$$

$$x = -\frac{1}{6}, -2$$

$$y = 2(-2)^3 + \frac{13}{2}(-2)^2 + 2(-2) + 4$$

$$y = -16 + 26 - 4 + 4$$

$$y = 10$$

$$(2, 10)$$

$$y = 2(-\frac{1}{6})^3 + \frac{13}{2}(-\frac{1}{6})^2 + 2(-\frac{1}{6}) + 4$$

$$y = \frac{829}{216}$$

$$(-\frac{1}{6}, \frac{829}{216})$$

Example 11: Find the derivative of each function

a. $y = 3x^{-6} - \frac{4}{5}x^{-4} + 2x^2 - 9$

$$y' = -18x^{-7} + \frac{16}{5}x^{-5} + 4x$$

b. $y = 4x^{-2} + 2x^{\frac{1}{5}} - 7x$

$$y' = -8x^{-3} + \frac{2}{5}x^{-4/5} - 7$$

c. $y = \frac{2}{3x^2} + \frac{5}{2x^7} + \sqrt[4]{3x^2} - \sqrt[7]{4x^9}$

$$y = \frac{2}{3}x^{-2} + \frac{5}{2}x^{-7} + \sqrt[4]{3}x^{1/2} - \sqrt[7]{4}x^{9/7}$$

$$y' = -\frac{4}{3}x^{-3} - \frac{35}{2}x^{-8} + \frac{1}{2}\sqrt[4]{3}x^{-1/2} - \frac{9}{7}\sqrt[7]{4}x^{2/7}$$

Day 2 The Product Rule

Now that you know how to find derivatives of simple polynomials, it's time to get more complicated. What if you had to find the derivative of this? $f(x) = (x^3 + 5x^2 - 4x + 1)(x^5 - 7x^4 + x)$

You could multiply out the expression and take the derivative of each term, like this:

$$f(x) = x^8 - 2x^7 - 39x^6 + 29x^5 + 6x^4 + 5x^3 + 4x^2 + x$$

And the derivative is:

$$f'(x) = 8x^7 - 14x^6 - 234x^5 + 145x^4 + 24x^3 + 15x^2 + 8x + 1$$

Needless to say, this process can get pretty messy. Naturally, there's an easier way. When a function involves two terms multiplied by each other, we use the **Product Rule**.

The Product Rule:

$$\text{If } f(x) = uv, \text{ then } f'(x) = \frac{du}{dx}v + \frac{dv}{dx}u$$

There is an easy saying to help remember this rule:

"The derivative of the first times the second plus the derivative of the second times the first."

Another way to write the Product Rule looks like this:

$$\text{The Product Rule: } \frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

Examples: Find dy/dx of each function.

1) $y = (2x + 3)(x^2 - 6)$

$$\begin{aligned}y' &= 2(x^2 - 6) + (2x)(2x + 3) \\ &= 2x^2 - 12 + 4x^2 + 6x \\ &= 6x^2 + 6x - 12\end{aligned}$$

2) $y = (\sqrt{x} + x)(2x^2 + 1)$

$$\begin{aligned}y' &= \left(\frac{1}{2}x^{-1/2} + 1\right)(2x^2 + 1) + (\sqrt{x} + x)(4x) \\ &= x^{3/2} + 2x^2 + \frac{1}{2}x^{-1/2} + 1 + 4x^{3/2} + 4x^2 \\ &= 6x^2 + 5x^{3/2} + \frac{1}{2}x^{-1/2} + 1\end{aligned}$$

3) $f(x) = (9x^2 + 4x)(x^3 - 5x^2)$

$$\begin{aligned}y' &= (18x + 4)(x^3 - 5x^2) + (9x^2 + 4x)(3x^2 - 10x) \\ &= 18x^4 + 4x^3 - 90x^3 - 20x^2 + 27x^4 + 12x^3 - 90x^3 - 40x^2 \\ &= 45x^4 - 164x^3 - 60x^2\end{aligned}$$

$$4) f(x) = (\sqrt{x} + 4\sqrt[3]{x})(x^5 - 11x^8)$$

$$\left(\frac{1}{2}x^{-1/2} + \frac{4}{3}x^{-2/3}\right)(x^5 - 11x^8) + (x^{1/2} + 4x^{1/3})(5x^4 - 88x^7)$$

$$\frac{1}{2}x^{9/2} + \frac{4}{3}x^{13/3} - \frac{11}{2}x^{15/2} - \frac{44}{3}x^{22/3} + 5x^{9/2} + 20x^{13/3} - 88x^{15/2} - 352x^{22/3}$$

$$f'(x) = \frac{11}{2}x^{9/2} + \frac{64}{3}x^{13/3} - \frac{187}{2}x^{15/2} - \frac{1100}{3}x^{22/3}$$

$$5) y = \left(\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}\right)\left(\frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5}\right)$$

$$= (x^{-1} + x^{-2} - x^{-3})(x^{-1} - x^{-3} + x^{-5})$$

$$= (-x^{-2} - 2x^{-3} + 3x^{-4})(x^{-1} - x^{-3} + x^{-5}) + (x^{-1} + x^{-2} - x^{-3})(-x^{-2} + 3x^{-4} - 5x^{-6})$$

$$-x^{-3} - 2x^{-4} + 3x^{-5} + x^{-5} + 2x^{-6} - 3x^{-7} - x^{-7} - 2x^{-8} + 3x^{-9} + (-x^{-3} - x^{-4} + x^{-5} + 3x^{-5} + 3x^{-6} - 3x^{-7} - 5x^{-7} - 5x^{-8} + 5x^{-9})$$

$$y' = -2x^{-3} - 3x^{-4} + 4x^{-5} + 4x^{-5} + 5x^{-6} - 6x^{-7} - 6x^{-7} - 7x^{-8} + 8x^{-9}$$

$$y' = -2x^{-3} - 3x^{-4} + 8x^{-5} + 5x^{-6} - 12x^{-7} - 7x^{-8} + 8x^{-9}$$

$$y' = x^{-9}(-2x^6 - 3x^5 + 8x^4 + 5x^3 - 12x^2 - 7x + 8)$$

For the following examples, write the equation of the tangent line at the given value of x .

6) $f(x) = (x^2 - 4x + 3)(x + 1)$ at $x = -1$.

$$\begin{aligned} f'(x) &= (2x - 4)(x + 1) + (x^2 - 4x + 3)(1) \\ &= 2x^2 - 4x + 2x - 4 + x^2 - 4x + 3 \\ &= 3x^2 - 6x - 1 \end{aligned}$$

$$\begin{aligned} f'(-1) &= 3(-1)^2 - 6(-1) - 1 \\ &= 8 \leftarrow \text{slope} \end{aligned}$$

Plug in x to find y

$$\begin{aligned} f(x) &= (-1)^2 - 4(-1) + 3(-1 + 1) \\ &= 0 \\ &(-1, 0) \end{aligned}$$

$$y - 0 = 8(x + 1)$$

$$\boxed{y = 8x + 8}$$

7) $f(x) = (\sqrt[3]{x} - 2x)(3x + 1)$ at $x = 8$.

$$\begin{aligned} f'(x) &= \left(\frac{1}{3}x^{-2/3} - 2\right)(3x + 1) + (x^{1/3} - 2x)(3) \\ &= x^{1/3} - 6x + \frac{1}{3}x^{-2/3} - 2 + 3x^{1/3} - 6x \\ &= 4x^{1/3} - 12x + \frac{1}{3}x^{-2/3} - 2 \end{aligned}$$

$$\begin{aligned} f'(8) &= 4(8)^{1/3} - 12(8) + \frac{1}{3}(8)^{-2/3} - 2 \\ &= \frac{-1079}{12} \end{aligned}$$

$$y + 350 = \frac{-1079}{12}(x - 8)$$

$$(8, -350)$$

8) $y = (5x^2 + x)(x^6 - 6x^5)$ at $x = 1$.

$$\begin{aligned} y' &= (10x + 1)(x^6 - 6x^5) + (5x^2 + x)(6x^5 - 30x^4) \\ &= 10x^7 + x^6 - 60x^6 - 6x^5 + 30x^7 + 6x^6 - 150x^6 - 30x^5 \\ y' &= 40x^7 - 203x^6 - 36x^5 \end{aligned}$$

$$\begin{aligned} f(1) &= 40(1)^7 - 203(1)^6 - 36(1)^5 \\ &= -199 \end{aligned}$$

$$\boxed{y + 30 = -199(x - 1)}$$

$$(1, -30)$$

Day 3: The Quotient Rule

What happens when you have to take the derivative of a function that is the quotient of two other functions?

The Quotient Rule:

$$\text{If } f(x) = \frac{u}{v} \text{ then } f'(x) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

This one looks much more complicated, but there is an easy saying to help remember this rule too:

"The derivative of the top times the bottom minus the derivative of the bottom times the top, all over the bottom squared."

Another way to write the Quotient Rule looks like this:

$$\text{Quotient Rule: } \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Examples: Find $\frac{dy}{dx}$ of each function.

1) $y = \frac{2x^2+1}{x+5}$

$$y' = \frac{(4x)(x+5) - (2x^2+1)(1)}{(x+5)^2}$$

$$= \frac{4x^2 + 20x - 2x^2 - 1}{(x+5)^2}$$

$$= \frac{2x^2 + 20x - 1}{(x+5)^2}$$

$$2) h(x) = \frac{\sqrt{x+1}}{3x^3}$$

$$h'(x) = \frac{(\frac{1}{2}x^{-1/2})(3x^3) - (x^{1/2}+1)(9x^2)}{(3x^3)^2}$$

$$= \frac{\frac{3}{2}x^{5/2} - 9x^{5/2} + 9x^2}{9x^6}$$

$$= \frac{-\frac{15}{2}x^{5/2} - \frac{18}{2}x^2}{9x^6}$$

$$\rightarrow = \frac{3x^2(-5x^{1/2}-6)}{2 \cdot 9x^6}$$

$$= \frac{3x^2(-5x^{1/2}-6)}{18x^6}$$

$$= \frac{-5x^{1/2}-6}{6x^4}$$

$$3) f(x) = \frac{x^5+3x^4}{x^2+7x}$$

$$f'(x) = \frac{(5x^4+12x^3)(x^2+7x) - (x^5+3x^4)(2x+7)}{(x^2+7x)^2}$$

$$= \frac{5x^6 + 35x^5 + 12x^5 + 84x^4 - 2x^6 - 7x^5 - 6x^5 - 21x^4}{(x^2+7x)^2}$$

$$= \frac{3x^6 + 34x^5 + 63x^4}{x^2(x+7)^2} = \frac{3x^4 + 34x^3 + 63x^2}{(x+7)^2}$$

$$4) f(x) = \frac{x^{-3}-x^{-8}}{x^{-2}+x^{-6}}$$

$$f'(x) = \frac{(-3x^{-4}+8x^{-9})(x^{-2}+x^{-6}) - (x^{-3}-x^{-8})(-2x^{-3}-6x^{-7})}{(x^{-2}+x^{-6})^2}$$

$$= \frac{-3x^{-6} - 3x^{-10} + 8x^{-11} + 8x^{-15} + 2x^{-6} + 6x^{-10} - 2x^{-11} - 6x^{-15}}{x^{-4} + 2x^{-8} + x^{-12}}$$

$$= \frac{2x^{-15} + 6x^{-11} + 3x^{-10} - x^{-6}}{x^{-4} + 2x^{-8} + x^{-12}}$$

$$= \frac{x^{-15}(2+6x^4+3x^5-x^9)}{x^{-12}(x^8+2x^4+1)}$$

$$\rightarrow \frac{-x^9 + 3x^5 + 6x^4 + 2}{x^{11} + 2x^7 + x^3}$$

Find the equation of the tangent line to the function at the given point.

5) $f(x) = \frac{(x+1)}{4x}$ at $x = -1$

$f(-1) = \frac{-1+1}{4(-1)} = 0 \quad (-1, 0)$

$$f'(x) = \frac{1(4x) - (x+1)(4)}{(4x)^2}$$

$$= \frac{4x - 4x - 4}{16x^2}$$

$$= \frac{-4}{16x^2} = -\frac{1}{4x^2}$$

$f'(-1) = \frac{-1}{4(-1)^2} = -\frac{1}{4}$ ← slope

$$y = -\frac{1}{4}x - \frac{1}{4}$$

6) $y = \frac{x^2-3}{x-3}$ at $x = 4$ $f(4) = \frac{4^2-3}{4-3} = \frac{13}{1} \Rightarrow (4, 13)$

$$y' = \frac{2x(x-3) - (x^2-3)(1)}{(x-3)^2}$$

$$= \frac{2x^2 - 6x - x^2 + 3}{(x-3)^2}$$

$$= \frac{x^2 - 6x + 3}{(x-3)^2}$$

$f(4) = \frac{4^2 - 6(4) + 3}{(4-3)^2} = -5$

$$y - 13 = -5(x - 4)$$

$$y = -5x + 33$$

7) $y = \frac{x^2+2x}{x+1}$ at $x = 2$ $f(2) = \frac{2^2+2(2)}{2+1} = \frac{8}{3} \Rightarrow (2, 8/3)$

$$y' = \frac{(2x+2)(x+1) - (x^2+2x)(1)}{(x+1)^2}$$

$$= \frac{2x^2 + 4x + 2 - x^2 - 2x}{(x+1)^2}$$

$$= \frac{x^2 + 2x + 2}{(x+1)^2}$$

$f(2) = \frac{2^2 + 2(2) + 2}{(2+1)^2}$

$$= \frac{10}{9}$$

$$y - 8/3 = \frac{10}{9}(x - 2)$$

$$y = \frac{10}{9}x + \frac{4}{3}$$

8) $f(x) = \frac{x^3 - \sqrt{x}}{3x}$ at $x=4$

$f(4) = \frac{4^3 - \sqrt{4}}{3(4)} = \frac{62}{12} = \frac{31}{6}$

$$f'(x) = \frac{(3x^2 - \frac{1}{2}x^{-1/2})(3x) - (x^3 - x^{1/2})(3)}{(3x)^2}$$

$$= \frac{9x^3 - \frac{3}{2}x^{1/2} - 3x^3 + 3x^{1/2}}{9x^2}$$

$$= \frac{6x^3 + \frac{3}{2}x^{1/2}}{9x^2}$$

$$\rightarrow f'(4) = \frac{6(4)^3 + \frac{3}{2}(4)^{1/2}}{9(4)^2}$$

$$= \frac{384 + 3}{144}$$

$$= \frac{43}{16}$$

$y - \frac{31}{6} = \frac{43}{16}(x - 4)$

Differentiate each function with respect to x.

9) $y = \frac{x^2 + 2x}{x + 1}$

Same as # 7

~~$$y' = \frac{(2x + 2)(x + 1) - (x^2 + 2x)(1)}{(x + 1)^2}$$

$$= \frac{2x^2 + 4x + 2 - x^2 - 2x}{(x + 1)^2}$$~~

~~10) $f(x) = \frac{x^3 - \sqrt{x}}{3x}$~~

Day 4: The Chain Rule

Compositions

Exploration: Find the derivative of each $f(x)$ and $g(x)$. Then find the composition $f(g(x))$.

1) $f(x) = x^2$

$$f'(x) = 2x$$

$$g(x) = 3x + 7$$

$$g'(x) = 3$$

$$f[g(x)] =$$

$$(3x+7)^2 = 9x^2 + 42x + 49$$

$$\text{Find } \frac{d}{dx}[f(g(x))] = 18x + 42$$

(using chain rule: $2(3x+7) \cdot 3$)

2) $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$g(x) = 6x$$

$$g'(x) = 6$$

$$f[g(x)] =$$

$$\sqrt{6x} = \sqrt{6}x^{1/2}$$

Use example 1 to find

$$\frac{d}{dx}[f(g(x))] = \frac{1}{2}x^{-1/2}\sqrt{6}$$

State the chain rule in symbols and words in the space provided below.

$(f'(g(x))) \cdot g'(x)$ work outside in.

Example 3: Use the chart below to find $f'(g(-2))$.

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
$x = -2$	-1	3	5	-7
$x = 3$	8	2	-2	4
$x = -7$	1	11	6	-9

$$f'(3)$$
$$f'(3) = -2$$

The most important rule in this chapter (and sometimes the most difficult) is called the **Chain Rule**. It's used when you're given composite functions--that is, a function inside of another function. These are tested heavily in Calculus; therefore, it is important to have the **Chain Rule** down cold.

A composite function is usually written as: $f(g(x))$.

For example: If $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{3x}$, then $f(g(x)) = \frac{1}{\sqrt{3x}}$

We could also find $g(f(x)) = \sqrt{\frac{3}{x}}$

When finding the derivative of a composite function, we take the derivative of the "outside" function, with the inside function g considered the variable, leaving the inside function alone. Then, we multiply this by the derivative of the "inside" function, with respect to x .

The Chain Rule:

$$\text{If } y = f(g(x)), \text{ then } y' = \left(\frac{df}{dg}(g(x)) \right) \left(\frac{dg}{dx} \right)$$

- You need to find the derivative of each.

Again, this looks much more complicated than it really is. The saying to help remember this one is:

"The derivative of the outside times the derivative of the inside."

Example 1: If $y = (5x^3 + 3x)^5$, then $\frac{dy}{dx} = 5(5x^3 + 3x)^4(15x^2 + 3)$

We just dealt with the derivative of something to the fifth power, like this:

$$y = (g)^5 \frac{dy}{dg} = 5(g)^4, \text{ where } g = 5x^3 + 3x$$

Then we multiplied by the derivative of g , $15x^2 + 3$.

Always do it this way. The process has several successive steps, like peeling away layer of an onion until you reach the center.

Example 2: Find $\frac{dy}{dx}$ for $y = \sqrt{x^3 + 4x} = (x^3 + 4x)^{1/2}$

$$y' = \frac{1}{2} (x^3 + 4x)^{-1/2} (3x^2 + 4)$$

$$y' = \frac{3x^2 + 4}{2(x^3 + 4x)^{1/2}}$$

Example 3: Find $\frac{dy}{dx}$ for $y = (2\sqrt{x} + 3x)^2$.

$$y' = 2(2x^{1/2} + 3x)(x^{-1/2} + 3)$$

$$y' = (4x^{1/2} + 6x)(x^{-1/2} + 3)$$

$$y' = 4 + 12x^{1/2} + 6x^{1/2} + 18x$$

$$y' = 18x + 18x^{1/2} + 4$$

Example 4: Use the chain rule and product rule to find $\frac{dy}{dx}$ for $y =$

$$\sqrt{(x^5 - 8x^3)(x^2 + 6x)} = (x^5 - 8x^3)^{1/2} (x^2 + 6x)^{1/2}$$

$$y' = \frac{1}{2}(x^5 - 8x^3)^{-1/2} (x^2 + 6x)^{1/2} (5x^4 - 24x^2) + (\frac{1}{2})(x^2 + 6x)^{-1/2} (x^5 - 8x^3)^{1/2} (2x + 6)$$

$$y' = \frac{1}{2} (x^5 - 8x^3)^{-1/2} (x^2 + 6x)^{-1/2} [(x^2 + 6x)(5x^4 - 24x^2) + (x^5 - 8x^3)(2x + 6)]$$

$$y' = \frac{1}{2} (x^5 - 8x^3)^{-1/2} (x^2 + 6x)^{-1/2} [5x^6 - 24x^4 + 30x^5 - 144x^3 + 2x^6 + 6x^5 - 16x^4 - 48x^3]$$

$$y' = \frac{1}{2} (x^5 - 8x^3)^{-1/2} (x^2 + 6x)^{-1/2} [7x^6 + 36x^5 - 40x^4 - 192x^3]$$

Example 5: Use the chain rule and the quotient rule to find $\frac{dy}{dx}$ for $y =$

$$\left(\frac{2x+8}{x^2-10x}\right)^5 = \frac{(2x+8)^5}{(x^2-10x)^5}$$

$$y' = \frac{5(2x+8)^4 (x^2-10x)^5 (2) - 5(x^2-10x)^4 (2x+8)^5 (2x-10)}{(x^2-10x)^{10}}$$

$$y' = \frac{10(2x+8)^4 (x^2-10x)^4 [x^2-10x - (x-5)(2x+8)]}{(x^2-10x)^{10-6}}$$

↙ lost the 2 (gcf)

$$y' = \frac{10(2x+8)^4 [x^2-10x-2x^2-8x+10x+40]}{(x^2-10x)^6} = \frac{10(2x+8)^4 (-x^2-8x+40)}{(x^2-10x)^6}$$

Example 6: Find $\frac{dy}{dx}$ at $x = 1$ if $y = [(x^3 + x)(x^4 - x^2)]^2$.

$$y' = 2(x^3 + x)(x^4 - x^2)^2(3x^2 + 1) + 2(x^4 - x^2)(x^3 + x)^2(4x^3 - 2x)$$

$$y' = 2(x^4 - x^2)(x^3 + x)[(x^4 - x^2)(3x^2 + 1) + (x^3 + x)(4x^3 - 2x)]$$

$$y' = 2(x^4 - x^2)(x^3 + x)[3x^6 + x^4 - 3x^4 - x^2 + 4x^6 - 2x^4 + 4x^4 - 2x^2]$$

$$y' = 2(x^4 - x^2)(x^3 + x)(7x^6 - 3x^2)$$

$$f'(1) = 2(1^4 - 1^2)(1^3 + 1)(7(1)^6 - 3(1)^2)$$

$$f'(1) = 2(1-1)(1+1)(7-3) = 0$$

Example 7: Find the equation of the tangent line for $y = (2\sqrt{x} + 3x)^2$ at

$$x = 4. \quad y = (2\sqrt{4} + 3(4))^2 = (4 + 12)^2 = 256$$

$$y' = 2(2x^{1/2} + 3x)(x^{-1/2} + 3)$$

$$y' = 4 + 12x^{1/2} + 6x^{1/2} + 18x$$

$$y'(4) = 18(4) + 18(4)^{1/2} + 4$$

$$y'(4) = 112$$

$$y - 256 = 112(x - 4)$$

Example 8: Differentiate $y = (2x^4 - 5)^3 \sqrt[4]{-6x^5 - 1}$ with respect to x

$$y' = 3(2x^4 - 5)^2(-6x^5 - 1)^{1/4}(8x^3) + \frac{1}{4}(-6x^5 - 1)^{-3/4}(2x^4 - 5)^3(-30x^4)$$

$$y' = 24x^3(2x^4 - 5)^2(-6x^5 - 1)^{1/4} - \frac{15}{2}x^4(-6x^5 - 1)^{-3/4}(2x^4 - 5)^3$$

$$y' = 6x^3(2x^4 - 5)^2(-6x^5 - 1)^{-3/4} [4(-6x^5 - 1) - \frac{5}{4}(2x^4 - 5)]$$

$$y' = 6x^3(2x^4 - 5)^2(-6x^5 - 1)^{-3/4} (-24x^5 - 4 - \frac{5}{2}x^4 + \frac{25}{4})$$

$$y' = 6x^3(2x^4 - 5)^2(-6x^5 - 1)^{-3/4} (-24x^5 - \frac{5}{2}x^4 + 2)$$